Monodromy groups of irregular elliptic surfaces

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Abstract

Monodromy groups, i.e. the groups of isometries of the intersection lattice $L_X := H_2/\text{torsion}$ generated by the monodromy action of all deformation families of a given surface, have been computed in [Lö'] for any minimal elliptic surface with $p_g > q = 0$. New and refined methods are now employed to address the cases of minimal elliptic surfaces with $p_g \geq q > 0$.

Thereby we get explicit families such that any isometry is in the group generated by their monodromies or does not respect the invariance of the canonical class or the spinor norm. The monodromy is also shown to act by the full symplectic group on the first homology modulo torsion.

Introduction

Monodromy is a powerful tool and has been a predominant subject of interest in the realms of singularity since long. This paper as its predecessor tries to broaden the view to include the case of deformation families of compact complex surfaces. This first step leaving the ground of surfaces singularities and complete intersection surfaces just reaches the case of elliptic surfaces but nevertheless it establishes an astonishing likeness between the topological notion of the representation of the diffeomorphism group and the analytical notion of the monodromy action. In fact the associated isometry groups are the same up to index two.

In the first section we construct families with monodromy acting on a simply connected part of a reference surface which is isomorphic to the Milnor fibre of a hypersurface singularity. But in contrast to the case of regular elliptic surfaces this Milnor fibre supports only classes generating a sublattice of L_X of possibly high corank.

So next we give families which yield vanishing cycles generating a sublattice containing the classes of Lagrangian tori in the bundle part of the reference surface.

Finally we have to take care again of the fibre classes which is done as in the regular case. The results thus collected fit together nicely and yield the main result:

Main Theorem *Let X be a minimal elliptic surface with positive holomorphic Euler number $\chi = p_g - q + 1$ and positive irregularity q, then there exist families of elliptic surfaces containing X, such that the induced monodromy actions on the homology lattice L_X generate $O'_k(L_X)$, the group of isometries of real spinor norm one fixing the canonical class.*

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Families obtained from unfoldings

The aim of the first section is to construct families of irregular elliptic surfaces, which contain Morsifications of isolated singularities of maximal possible Milnor numbers. Let us start with the Hirzebruch surfaces $\mathbf{F}_1, \mathbf{F}_2$ ruled over $\mathbf{P}^1$ containing sections $\sigma_1^\infty, \sigma_2^\infty$ of negative square and with branch divisors B_1, B_2 resp. given by

$$
\begin{aligned}
B_1 &= \sigma_1^\infty \cup \overline{\{x(y^3 - 3y - x^3) = 0\}}, \\
B_2 &= \sigma_1^\infty \cup \overline{\{y^3 - 3y - x^6 = 0\}}.
\end{aligned}
$$

The corresponding double covers are rational elliptic surfaces which shall be denoted by $X_{1/2}, X_1$ respectively.

Generalizing this construction we define $\mathcal{Z}_\chi$ to be the family of elliptic surfaces given by the family of double covers of $\mathbf{F}_{2\chi}$ along the family of branch curves consisting of the curves given by $y^3 - 3y - x^{6\chi} - \lambda x - u$ together with the negative section:

$$
\begin{array}{ccc}
\mathcal{Z} & \to & \mathbf{P}^1 \times \mathbf{C}^2 \\
& \searrow \quad \swarrow & \\
& \mathbf{C}^2 \quad \ni (\lambda, u) &
\end{array}
$$

The negative section lifts to $\sigma_{\mathcal{Z}}$ which restricts to a section of each surface of $\mathcal{Z}$.

Note that incidentally for $\chi > 1$ the surface Z_0 in $\mathcal{Z}_\chi$ with parameters $\lambda = u = 0$ coincides with the pull back of X_1 along the degree χ cyclic Galois cover of $\mathbf{P}^1$ branched at zero and infinity. Similarly if we take the pull back of $X_{1/2}$ along the double cover of $\mathbf{P}^1$ branched at zero and infinity we have to normalize and blow down exceptional divisors in the fibres to get the central surface Z_0 of $\mathcal{Z}_1$.

Lemma 1 *There is an open polycylinder $S \subset \mathbf{C}^2$ containing the origin and an open holomorphic disc D with $\infty \in D \subset \bar{D} \subset \mathbf{P}^1 \setminus \{0\}$ such that*

> i) *for all $s \in S$ the branch points of the Kodaira j-function j_s on $\mathbf{P}^1$ associated to Z_s are contained in $D \cup \{0\}$,*

> ii) *for all $s \in S$ the preimage $U_s := j_s^{-1}(D)$ is a holomorphic disc which covers D with degree 12χ,*

> iii) *the induced family $\mathcal{Z}'$ of elliptically fibred surfaces over varying holomorphic discs*

$$
\begin{array}{ccc}
\mathcal{Z}' & \to & \{U_s\}_{s \in S} \subset \mathbf{P}^1 \times S \\
& \searrow \quad \swarrow & \\
& S &
\end{array}
$$

> *contains a Morsification of a singularity of type $J_{2\chi}$.*

Proof: The j-function can explicitly be given in terms of the base coordinate x and the parameter $s = (u, \lambda)$ as

$$j_s(x) = \frac{1}{1 - 4(x^{6\chi} + \lambda x + u)^2}.$$

The formula shows that the associated maps from $\mathbf{P}^1$ to itself are of degree 12χ with no zero except at infinity. Hence zero is a branch point of constant multiplicity and does not vary.

On the other hand the branching in the neighbourhood of infinity does vary, but since branch points vary continuously with the parameters, their range is bounded as soon as the parameters are. Thus for any bounded polydisc S a disc D can be found as claimed such that $i)$ holds.

To get $ii)$ we argue with the disc D^c complementary to D in $\mathbf{P}^1$. Since j_s is totally ramified at zero with no other branching in D^c it is equivalent to the standard branching $z \mapsto z^{12\chi}$, and the preimage is therefore a disc. And so is its complement, the cover U_s of D.

Finally we observe that in the family $f_\lambda = z^2 + y^3 - 3y - x^{6\chi} - \lambda x$ Morse functions are dense, cf. [AGLV], and that any bound on λ will imply a uniform bound on the range of the critical values of the f_λ. Choose S accordingly to be any polycylinder given by such a pair of bounds. Then by construction $\mathcal{Z}|_S$ contains a Morsification of the singularity of type $J_{2\chi}$ given by the local equation $z^2 + y^3 = x^{6\chi}$. Since each surface in $\mathcal{Z}|_S$ fibres smoothly over $\mathbf{P}^1 \setminus U_s$, also $\mathcal{Z}'$ contains the Morsification and so $iii)$ holds. $\square$

The construction of a reference surface uses suitable branched covers:

Lemma 2 *Given positive integers $n \geq 2, q$ and a set $\{x_0, ..., x_{2q+n}\}$ of distinct points of $\mathbf{P}^1$, then there is a curve C_q of genus q and a branched covering $\pi : C_q \to \mathbf{P}^1$ of degree n totally ramified at x_1 such that the associated monodromy representation*

$$\rho : \pi_1(\mathbf{P}^1 \setminus \cup_{i>0}\{x_i\}, x_0) \to \mathbf{S}_n$$

assigns the following permutations to simple loops ω_i associated to a given geometric basis of paths from x_0 to the x_i:

$$
\begin{aligned}
\rho([\omega_j]) &= (12) & j = 2, ..., 2q+2 \\
\rho([\omega_{2q+j}]) &= (j{-}1 \ \ j) & j = 3, ..., n.
\end{aligned}
$$

Proof: The given monodromy representation assigns to the composition of the loops ω_i, $i > 1$ an element of order n. Hence the unramified cover of $\mathbf{P}^1 \setminus \{x_1\}$ determined by the subgroup $\rho^{-1}(\mathbf{S}_{n-1})$ of π_1 can be compactified by a single point to yield a branched covering $\pi : C_q \to \mathbf{P}^1$ as claimed. $\square$

Proposition 1 *Given discrete invariants $\chi, q, \mu = m_1, ..., m_n$ there is a smooth elliptic surface X with the given invariants which contains the central surface Z_0' of $\mathcal{Z}_\chi' \to S$.*

Proof: Let us first consider the case $\chi > 1$: Given any simply connected neighbourhood U of $x_1 = 0$ in $\mathbf{P}^1$ there is as in lemma 2 a cover $\pi : C_q \to \mathbf{P}^1$ of degree χ and genus q totally branched at zero with the additional property that all other branch points are in the complement of U. Then U is covered by a disc $\tilde{U}$ in C_q with total ramification at zero, just as in the restriction of the cyclic cover $c : \mathbf{P}^1 \to \mathbf{P}^1$ of degree χ branched at $\{0, \infty\}$ into which $\tilde{U} \to U$ thus necessarily embeds.
If we let X' be the pull back of X_1 along π and choose U to contain $c(U_0)$ then Z_0' embeds holomorphically into $X'|_{\tilde{U}}$ as follows from the observation made above. This property is not affected by the logarithmic transformations on X' necessary to get X if they are performed over the complement of $\tilde{U}$.
In the case of $\chi = 1$ we let X' instead be the minimal model of the normalization of the pull back of $X_{1/2}$ along a hyperelliptic cover π with branching outside U except for the branch point at zero. $\qquad\square$

Lemma 3 *Let Z_s and X be elliptic surfaces as introduced above, then there is an annulus A obtained from D as given by lemma 1 by removing a closed disc such that*

   i) $A_s := j_s^{-1}(A)$ is an annulus for all $s \in S$ where j_s is the j-invariant on $\mathbf{P}^1$ associated to the elliptic surface Z_s,

   ii) $A_X := j_X^{-1}(A)$ is an annulus for the j-invariant j_X on C_q associated to the elliptic surface X.

Proof: If A is chosen suitably it contains no branchpoints for all j_s and j_X and thus is unramified covered by A_s resp. A_X. The coverings are connected since the boundary coverings $\partial U_s \to \partial D$ are connected by lemma 1 and $\partial U_X \to \partial D$, where $U_X = j_X^{-1}(D)$, is equivalent to $\partial U_0 \to \partial D$ by construction of X, hence the claim. $\qquad\square$

Lemma 4 *There is an embedding $h = (h_1, h_2)$ of the subset $A_S := \cup_s A_s$ of $\mathbf{P}^1 \times S$ into $C_q \times S$ such that*

$$j_X(h_1(a, s)) = j_s(a) =: j_S(a, s), \quad \forall (a, s) \in A_S.$$

Proof: Both A_S and $A_X \times S$ are connected unramified covers of $A \times S$ of degree 12χ via j_S resp. $j_X \times \mathrm{id}_S$. Since the fundamental group of $A \times S$ is cyclic both

4

covers are homotopically equivalent and therefore holomorphically equivalent. The embedding is thus obtained by an equivalence followed by the obvious embedding of $A_X \times S$ into $C_q \times S$. $\square$

Lemma 5 *There is a unique holomorphic embedding $\tilde{h}$ of $\mathcal{Z}|_{A_S}$ into $X \times S$ covering h which maps points of $\sigma_{\mathcal{Z}}$ into $\sigma_0 \times S \subset Z_0' \times S \subset X \times S$.*

Proof: The functional invariant associated to ∂U_s varies continuously with $s \in S$ and thus is constant. The functional invariant along ∂U_X is the same, for it is equal to that of ∂U_0 by construction of X. Hence the functional invariant and the j-invariant by lemma 4 coincide on A_S and $A_X \times S$, and we get with thm. I.3.19 of [FM] a unique embedding $\tilde{h}$ of the fibrations covering h which respects the sections. $\square$

By discarding a suitable neighbourhood of the singular fibres in $X \times S$, the map $\tilde{h}$ becomes the biholomorphic identification of collars. By gluing we thus get a family $\mathcal{F}_{\chi,q}^{\mu} \to S$ of elliptic surfaces.
Notice that X is biholomorphic to the central surface and let $\mathcal{U}$ denote the open subset of $\mathcal{F}$ given by $\mathcal{Z}' \setminus \sigma_{\mathcal{Z}}$.

Our final aim in this paragraph is to describe the algebraic monodromy group $\Gamma(\mathcal{F}) \subset O(L_X)$ of the family $\mathcal{F}$ in terms of a set Δ of classes of square -2 as the group Γ_Δ generated by the reflections on hyperplanes normal to elements in Δ.

Proposition 2 *The monodromy group $\Gamma(\mathcal{F})$ of the smooth surface X in the family $\mathcal{F}$ coincides with the group Γ_{Δ_M} associated to the set Δ_M of all classes of square -2 contained in the sublattice $L_M \subset L_X$ generated by cycles in $M = X \cap \mathcal{U}$.*

Proof: As in the complement of $\mathcal{U}$ the family projection is a fibre bundle by construction, the monodromy acts only on L_M. We made sure that $\mathcal{U}$ contains a Morsification for the hypersurface singularity $z^2 + y^3 + x^{6\chi}$, so the set of vanishing cycles determines generators for the monodromy group via the associated reflections. In [E] this set is shown to coincide with the set of classes of square -2 in the Milnor lattice of the singularity which is just L_M and the claim follows. $\square$

Families obtained from branched coverings

In this section we construct families by varying the branch locus in our pull back construction. To understand the associated vanishing cycles we consider first the situation locally on the base, annuli covering a disc with varying branch locus:

Lemma 6 *Let Y_0 be smooth and properly fibred over a holomorphic disc V such that all fibres are smooth elliptic curves except for a singular fibre of type I_1. If Y is the pull back of Y_0 along a double cover $d : A \to V$ such that A is an annulus and the two branch points differ from the base of the singular fibre of Y_0, then*

   i) Y is properly elliptic fibred over A with two singular fibres of type I_1,

   ii) Y is diffeomorphic to a trivial torus bundle over A with a 2-handle added on each boundary with framing -1 along isotopic vertical curves,

   iii) $H_2(Y)$ with the intersection pairing is a lattice isomorphic to the diagonal lattice $(-2) \oplus (0)_{2g+1}$.

   Proof: The first assertion is obvious by base change properties. To get the second, we decompose Y_0 as a trivial torus bundle with a 2-handle added on a vertical loop with framing -1, cf. [K]. The pull back of the torus bundle yields the trivial bundle over an annulus, whereas the additional 2-handle lifts to handles which are added as claimed.
The trivial bundle supports the radical of the intersection lattice of rank $2g + 1$. The distinguished handles cancel with the same 1-handle, hence their cores glue to an embedded sphere of self-intersection -2. □

Lemma 7 *Let Y_0 be a smooth complex surface properly elliptic fibred over a holomorphic disc V with a single singular fibre of type I_1 at the origin, let*

$$d : \mathcal{A}_T \to V \times T$$

be a versal family of double covers of U with branch divisor of degree two, and $\mathcal{Y}_T$ be the pull back along d of the trivial family $Y_0 \times T$.
Then for any smooth surface Y in $\mathcal{Y}_T$ there are spheres $s_{\pm}^2$ embedded in Y, such that

   i) $s_{\pm}^2$ are vanishing cycles for ordinary double point degenerations in $\mathcal{Y}_T$,

   ii) $s_{\pm}^2$ map to simple arcs in the base, the union of which is isotopic to the core of the annulus,

   iii) the intersection of $s_{\pm}^2$ with a regular fibre is either empty or a vanishing cycle for both curve degenerations,

iv) the intersection $s_+^2 \cdot s_-^2$ is ± 2 with the sign depending on orientations,

v) dual classes to $s_\pm^2$ can be represented by cylinders mapping to a cocore of the annulus.

Proof: It suffices to give some family of double covers with the claimed properties. This family is obtained by pull back from any versal family, hence the versal family has the properties as well.

Consider first the family of double covers of the unit disc D_1 determined up to Galois involution by the branch locus $B_t = \{(2t + 1)/3, (2t - 1)/3\}$:

$$
\begin{array}{ccc}
\mathcal{A} & \rightarrow & D_1 \times D_1 \ni t, t' \\
& \searrow \quad \swarrow & \\
& D_1 \ni t &
\end{array}
$$

By a suitable biholomorphic equivalence we may regard V as the unit disc with the nodal curve of Y_0 at the origin.

Let $\mathcal{Y}$ be the family of fibred surfaces obtained by pull back of $Y_0 \times D_1$ along $\mathcal{A} \rightarrow D_1 \times D_1$. Then $\mathcal{Y}$ has two ordinary double point degenerations at $t = \pm\frac{1}{2}$. The associated vanishing cycles can be represented by spheres $s_\pm^2$ in the following way: Let vanishing cells for the curve degeneration of Y_0 be given over the segments $t = [0, \pm\frac{1}{2}]$ with a single transversal intersection. Then the pull back of the cells to $\mathcal{Y}$ are vanishing cells for the surface degeneration and yield spheres in Y as claimed in $i), ii), iii), iv)$. Also $v)$ is then immediate. □

Lemma 8 *If t^2 is a tube (cylinder) embedded into Y_0 which maps to a path in the base connecting the points $s = \pm\frac{1}{3}$ then its preimage $\tilde{t}^2$ in Y is an embedded torus which represents a class in the integer span of the classes of $s_\pm^2$ if the boundary circles of t^2 are vanishing cycles for the curve degeneration of Y_0.*

Proof: The preimage of such a tube consists of two tubes with boundary circles identified, a torus which represents an isotropic class in $H_2(Y)$ supported on the bundle part, cf. lemma 6.

By lemma 7 the sum of the classes represented by $s_\pm^2$ suitably oriented represent an isotropic class. Since the only horizontal torus in the bundle part which has the same intersection with all cylinders over a fixed cocore of the base annulus is the torus with vertical loop isotopic to the vanishing cycle of the curve degeneration, we are done. □

Getting on to the global setting we define for each set of discrete invariants χ, q, μ a family $\mathcal{E}_{\chi,q}^\mu$ of elliptic surfaces of the given invariants:

$\chi \geq 2$: Let $u_{\chi,q} : \mathcal{C} \to \mathbf{P}^1 \times H_{\chi,q}$ be the universal family of branched covers of $\mathbf{P}^1$ of degree χ and genus q with simple branching only except for a totally ramified branching at zero parametrized by the appropriate Hurwitz space $H_{\chi,q}$.

Let $u'_{\chi,q}$ denote the restriction of $u_{\chi,q}$ to the open part $H'_{\chi,q}$ parametrizing covers with no branching in an arbitrary small neighbourhood U^∞ of infinity. The family $\mathcal{E}_{\chi,q}$ is then defined to be the pull back of X_1 along $pr_1 \circ u'_{\chi,q}$. Since no branching is allowed on U^∞ the family is trivial over the preimage, so multiple fibres can be introduced there simultaneously to yield $\mathcal{E}^\mu_{\chi,q}$ for any given discrete invariants.

$\chi = 1$: Consider $u_{2,q}$ again, the universal family of hyperelliptic covers of genus q branching at zero and a varying set of further $2q+1$ distinct points of $\mathbf{P}^1 \setminus \{0\}$. Denote by $u''_{2,q}$ the restriction to the part H'' of the corresponding Hurwitz space which parametrizes covers with no branching in a small neighbourhood U^∞ of infinity nor in a small pointed neighbourhood U^0_* of zero. We get a family $\bar{\mathcal{E}}_q$ as the pull back of $X_{1/2}$ along $pr_1 \circ u'_q$. By the triviality over the preimage of U^∞ logarithmic transforms can again be performed simultaneously. The triviality over the preimage of $U^0 = U^0_* \cup \{0\}$ on the other hand makes it possible to normalize and blow down simultaneously and so we end up with a family $\mathcal{E}^\mu_q$ of elliptic surfaces with the preassigned invariants.

The choices involved in the constructions of $\mathcal{E}$ and X allow to assume that the surface X is contained in $\mathcal{E}$, because U^∞ can be chosen small enough to exclude the branch points x_i involved in the construction of X and the logarithmic transforms on X' and $\mathcal{E}_{\chi,q}$ can be performed in a compatible way too.

We may assume that a base point x_0 in $\mathbf{P}^1$ is chosen such that the fibre f_0 in $X_1, X_{1/2}$ respectively is smooth. Then there is a pair of vanishing cells projecting to paths p^+, p^- with associated vanishing cycles in f_0 dual to each other. On the base of the elliptic surface $X_1(X_{1/2})$ we may now choose the subsets considered above even more specifically:

i) a neighbourhood U^∞ of infinity not containing any singular values for the fibration map,

ii) a simply connected neighbourhood U as in prop. 1,

iii) points $x_j, j \geq 2$ ordered on a circle on $\mathbf{P}^1$ of constant modulus and a corresponding geometrically distinguished system of paths p_j in the complement of $U \cup U^\infty$,

iv) the base point x_0 on $\partial \bar{U}$

v) paths p^+, p^- contained in $\bar{U}$

In addition we will consider a collection $V_j^+, V_j^-, j = 2, ..., 2q + 2$ of holomorphic images of a disc such that V_j^+, V_j^- contain the circle segment between x_j and x_{j+1}, the paths p_j, p_{j+1} and p^+, resp. p^-. Moreover they are chosen to be disjoint from the singular values and the subset U^∞.
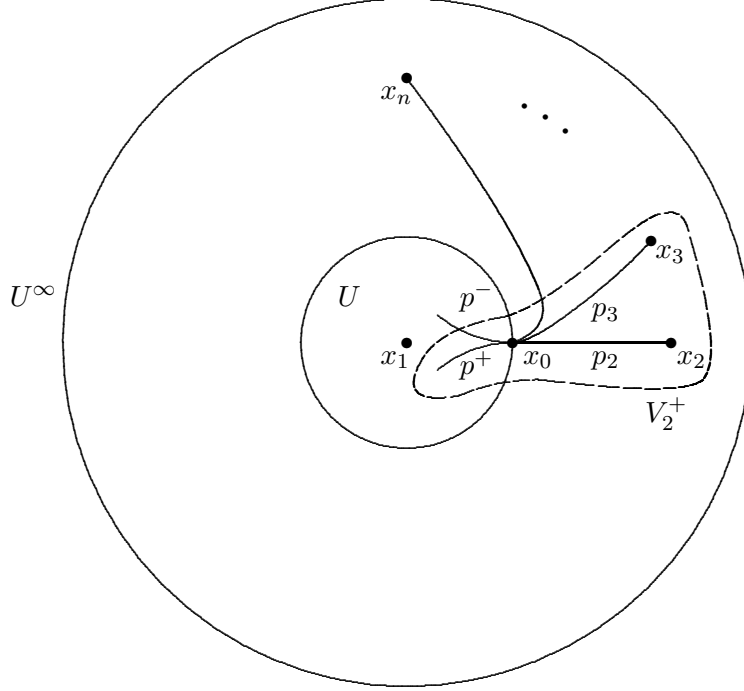
The vanishing cycles associated to these discs by lemma 7 then define a set Δ^b of classes of square -2 for X.

Lemma 9 *Let $\Gamma_{\chi,q}$ be the image of the monodromy representation of $\mathcal{E}_{\chi,q}^\mu$ on the homology lattice L_X^2 of X fibred over C_q^X and let Z^c be the complement of Z_0' in X. Then with the subset Δ^b of L_X^2 of elements of square -2*

i) *Γ_{Δ^b} is a subgroup of $\Gamma_{\chi,q}$,*

ii) *$H_2(Z^c)$ maps into the sublattice generated by Δ^b and the fibre classes.*

Proof: The family $u'_{\chi,q}$ (resp. $u''_{2,q}$) contains a versal family of double covers of each disc $V_j^\pm$ and we may apply lemma 7 to see that all elements of Δ^b are in fact vanishing cycles for ordinary double point degenerations of X in $\mathcal{E} = \mathcal{E}_{\chi,q}^\mu$.
The tubes over the circle segments with the vanishing cycles as vertical components lift thus to tori representing classes in the integer span of the vanishing classes of the family $\mathcal{E}$. It may be checked that these tori form a basis of a unimodular lattice of rank $2q$ and thus together with the fibre classes generate $H_2(Z^c)$. $\qquad\qquad \square$

Lemma 10 *Let Δ^s be the set of -2 classes of L_X supported on $M = Z_0 \cap \mathcal{U}$ and Δ the union $\Delta^s \cup \Delta^b$ then*

i) *Δ generates together with the fibre classes the orthogonal complement of the fibre class in L_X,*

ii) *Δ is contained in a single Γ_Δ orbit.*

Proof: Consider the Mayer-Vietoris sequence associated to the decomposition $X = Z^c \cup Z'_0$. We get a map $H_2(Z^c) \oplus H_2(Z'_0) \to H_2(X)$ with a torsionfree cokernel of rank one, which can be identified with the intersection with the class of a general fibre. Thus the sublattice orthogonal to the fibre classes is the image. Since Δ^b and the fibre classes generate $H_2(Z^c)$ and on the other hand Δ^s and the fibre classes generate $H_2(Z'_0)$ claim i) follows.
By construction each $\delta \in \Delta^b$ intersects algebraically trivial with the tori associated to the same vanishing cycle in f_0. They span a maximal isotropic sublattice, so δ is the orthogonal sum of an isotropic element in the sublattice spanned by the tori and an element of the orthogonal complement which is supported in Z'_0 and of square -2. Hence it differs at most by a multiple of the fibre class from an element in Δ^s. Hence there is another element δ' in Δ^s with $\delta.\delta' = 1$. So each element of Δ^s is conjugated to one of Δ^s since $\gamma_\delta \circ \gamma'_\delta(\delta) = \delta'$ for $\delta, \delta' \in \Delta$ with $\delta.\delta' = 1$. The claim ii) now holds since Δ^s is even a single orbit of the Γ_{Δ^s} action. $\qquad\qquad\square$

Proposition 3 *Let Γ_X be the image of the monodromy representation for the surface X. Then there is a set Δ' of homology classes of square -2 and a sublattice $L' < L_X$ containing a unimodular one of corank two, such that L' and the fibre classes generate the orthogonal complement of the fibre class, $\Gamma_{\Delta'}$ is a subgroup of Γ_X and L', Δ' is a complete vanishing lattice in the sense that, cf. [E, 5.3.1]:*

i) *Δ' generates L',*

ii) *Δ' is a single $\Gamma_{\Delta'}$ orbit,*

iii) *Δ' contains six elements the intersection diagram of which is*
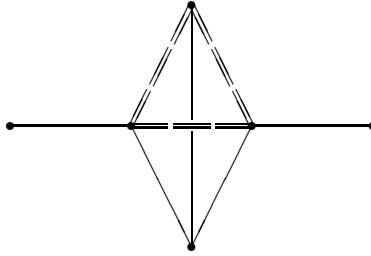
Proof: Let Δ' be $\Gamma_\Delta \cdot \Delta$ then Δ' generates a sublattice L' as claimed and $i), ii)$ are obvious from previous results, as is $iii)$ in case $\chi \geq 2$.

For $\chi = 1$ we note that Δ^s generates a sublattice of type E_8, hence Δ^s and either half of Δ^b – corresponding to the superscript '+' or '−' – generate a semidefinite lattice which meets the hypotheses of [FM, II.5.9]. We may conclude that with any element of Δ^s there are all elements of L' in Δ' which differ from the given by only a torus as considered in the proof of lemma 9.

The first four of them $\tilde{t}_1^+, \tilde{t}_1^-, \tilde{t}_2^+, \tilde{t}_2^-$ intersect like a symplectic basis with the only non-zero intersection being $\tilde{t}_1^+ . \tilde{t}_2^- = -\tilde{t}_1^- . \tilde{t}_2^+ = 1$. With elements $\alpha_1, \alpha_2, \alpha_3, \beta$ of Δ^s orthogonal except for $\alpha_1 . \beta = 1$ the following six elements are in Δ' and intersect as in $iii)$:

$$\alpha_2 + \tilde{t}_1^+, \alpha_1 + \tilde{t}_2^-, \alpha_1, \beta, \alpha_1 + \tilde{t}_1^-, \alpha_3 - \tilde{t}_2^+.$$

$\square$

Proposition 4 *Let X be a minimal elliptic surface with $p_g \geq q > 0$. Then the set Δ of classes in L_X of square -2 orthogonal to the fibre class f and represented by a vanishing cycle in a degeneration family of X generates the orthogonal complement L of the fibre class in L_X and $L, \Gamma_\Delta \cdot \Delta$ is a complete vanishing lattice.*

Proof: As in [Lö'] the proof relies on the deformation equivalence through elliptic surfaces of any surface of specified discrete invariants to our reference surface and on two further ingredients, a complete vanishing lattice as given by the previous proposition and for each fibre class a pair of vanishing cycles the sum of which is represented by the fibre.

The latter is obtained by a slight generalization to the irregular case of results proved in [Lö']. $\square$

Monodromy results

Monodromy defines a group of isotopy classes of orientation preserving diffeomorphisms. This monodromy group has natural representations on $L_X^2 := H_2/\text{torsion}$ and on $L_X^1 := H_1/\text{torsion}$.

The image of the first is contained in the group $O(L_X^2)$ of orthogonal transformations of L_X^2 with respect to the symmetric intersection pairing q_X. In case of an irregular elliptic surface we have an analogous property for the other representation:

Lemma 11 *Let $X \to C$ be an irregularly fibred minimal elliptic surface with positive Euler number. Then the induced map on homology $H_1(X) \to H_1(C)$ factors through an isomorphism*

$$L_X^1 \xrightarrow{\cong} H_1(C),$$

and the pull back of the intersection product on $H_1(C)$ is a skew-symmetric non-degenerate bilinear form b_X on L_X^1 such that the image of the natural monodromy representation is contained in the group of symplectic transformations

$$Sp(L_X^1) := Sp(L_X^1, b_X).$$

Proof: The fibration map is the Albanese mapping of X, hence induces an isomorphism on the first rational homology. Since the integer homology of C is torsionfree, the first claim follows.

Furthermore the bilinear form b_X is then skew-symmetric and non degenerate.

Finally each isotopy class of monodromy diffeomorphisms is represented by a fibration preserving one, hence commutes with a diffeomorphism of the base. The claim will follow as soon as the latter is shown to be orientation preserving.

If it were orientation reversing, so would be the action on a homology class of a multisection. On the other hand this contradicts the fact that fibre class and intersection product are preserved under monodromy, hence the claim. $\square$

Theorem 5 *Let X be a minimal elliptic surface with $p_g \geq q > 0$ and fibre class f. Then the group of orthogonal transformations of its homology lattice L_X^2 generated by the monodromy groups of all families containing X is*

$$O_f'(L_X) := \{\gamma \in O(L_X^2) | \gamma(f) = f, \gamma \text{ has positive real spinor norm}\}.$$

Proof: Again the argument is the same as in [Lö'] provided that the ingredients of the proof are extended to the irregular case, as we summarized in prop. 4. $\square$

Corollary 1 *Let X be as above. Then $O_f'(L_X^2)$ is even generated by elements associated to monodromy transformations acting trivially on L_X^1.*

Proof: Any diffeomorphism isotopic to the inversion at a -2 sphere acts trivially on L_X^1, hence the claim follows from the proof of the theorem. $\square$

Theorem 6 *Let X be as above. Then the group of transformations of L_X^1 generated by the monodromy groups of all families containing X is*

$$Sp(L_X^1).$$

Proof: Let $\mathcal{C}$ be a family of curves of genus q with the full symplectic monodromy. Then a family of elliptic surfaces deformation equivalent to X is obtained by the following construction: Choose a divisor on $\mathcal{C}$ disjoint from the critical locus of fibre degree 2χ without horizontal component (a 2χ-section). Take modulo a suitable base change the corresponding double cover $\tilde{\mathcal{C}}$ and associate the family $\tilde{\mathcal{C}}$ of trivial elliptic surfaces to it. Divide out the diagonal action of the Galois action on $\tilde{\mathcal{C}}$ and the involution on E and resolve the singularities of the quotient. Get thus the desired family of elliptic surfaces over the base of $\mathcal{C}$. Multiple fibres can again be introduced.
The resulting monodromy of the surface acts surely by Sp on the first homology of the base and hence on L_X^1 by the isomorphism of lemma 11. $\qquad\square$

Corollary 2 *Let X be as above. Then the monodromy representation on $L_X^1 \oplus L_X^2$ maps onto*

$$O_f'(L_X^2) \times Sp(L_X^1).$$

Proof: Let γ be any element of $O_f'(L_X^2) \times Sp(L_X^1)$, then we can find γ^1 acting on L_X^1 as γ does. Moreover γ^1 acts on L_X^2 as an element of O_f', hence $\gamma \circ (\gamma^1)^{-1}$ is some element of $O_f' \times \{id\}$ which is given by a monodromy element according to cor. 1. $\qquad\square$

Remark: In [Lö] the image of the natural representation of the diffeomorphism group of an elliptic surface as above on the homology lattice was shown to be generated by $O_f'(L_X)$ and a diffeomorphism σ – induced by complex conjugation – such that $O_f'(L_X)$, which is now shown to coincide with the monodromy group, is a subgroup of index two.

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